

**SHARP LYAPUNOV'S INEQUALITY FOR THE
MEASURABLE SETS WITH INFINITE MEASURE,
with generalization to the Grand Lebesgue spaces.**

E.Ostrovsky^a, L.Sirota^b

^a Corresponding Author. Department of Mathematics and computer science,
Bar-Ilan University, 84105, Ramat Gan, Israel.

E-mail: eugostrovsky@list.ru

^b Department of Mathematics and computer science. Bar-Ilan University, 84105,
Ramat Gan, Israel.

E-mail: sirota3@bezeqint.net

ABSTRACT.

We extend the classical Lyapunov inequality on the measurable space with infinite measure and on the so-called Grand Lebesgue spaces (GLS).

We find also the exact value for correspondent constant.

Possible applications: Functional Analysis (for instance, interpolation of operators), Integral Equations, Probability Theory and Statistics (tail estimations for random variables) etc.

Key words and phrases: Lebesgue-Riesz and Grand Lebesgue spaces (GLS), measurable set, double ratio between rearrangement invariant spaces, fundamental function, upper and lower bounds, rearrangement invariant space and norm, localized GLS norm, measurable function (random variable), Lyapunov's inequality.

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1 Notations. Statement of problem.

Let $(X = \{x\}, \mathbf{B}, \mu)$ be measurable space with non-trivial sigma-finite measure μ . We denote as ordinary for arbitrary measurable numerical (or complex valued) function $f : X \rightarrow \mathbb{R}$ and for any measurable set $A : A \in \mathbf{B}$ with finite non-zero measure: $0 < \mu(A) < \infty$

$$|f|_{p,A} = \left[\int_A |f(x)|^p d\mu(x) \right]^{1/p}, \quad (1.1)$$

$$|f|_p = \left[\int_X |f(x)|^p d\mu(x) \right]^{1/p}, \quad 1 \leq p < \infty,$$

$$f \in L_p(A) \Leftrightarrow |f|_{p,A} < \infty, \quad f \in L_p \Leftrightarrow |f|_p < \infty.$$

Note that $|f|_p = |f|_{p,X}$.

The classical Lyapunov's inequality asserts that if $\mu(X) = 1$, $1 \leq p \leq q$, then $|f|_p \leq |f|_q$. Therefore under at the same restrictions on the variables p, q there holds the following inequality

$$\frac{|f|_{p,A}}{\mu^{1/p}(A)} \leq \frac{|f|_{q,A}}{\mu^{1/q}(A)}, \quad 0 < \mu(A) < \infty. \quad (1.2)$$

Our purpose in this short report is to generalize the Lyapunov's inequality (1.2) into the so-called Grand Lebesgue Spaces (GLS) instead the classical Lebesgue-Riesz spaces constructed over the measurable space with (in general case) infinite measure.

We must recall here briefly the definition and some simple properties of the so-called Grand Lebesgue spaces; more detail investigation of these spaces see in [3], [4], [6], [11], [15], [16]; see also reference therein.

Recently appear the so-called Grand Lebesgue Spaces $GLS = G(\psi) = G\psi = G(\psi; a, b)$, $a, b = \text{const}, a \geq 1, a < b \leq \infty$, spaces consisting on all the measurable functions $f : X \rightarrow R$ with finite norms

$$\|f\|G(\psi) \stackrel{\text{def}}{=} \sup_{p \in (a,b)} [|f|_p / \psi(p)]. \quad (1.3)$$

Here $\psi(\cdot)$ is some continuous positive on the *open* interval (a, b) function such that

$$\inf_{p \in (a,b)} \psi(p) > 0, \quad \psi(p) = \infty, \quad p \notin (a, b).$$

We will denote

$$\text{supp}(\psi) \stackrel{\text{def}}{=} (a, b) = \{p : \psi(p) < \infty, \}$$

The set of all ψ functions with support $\text{supp}(\psi) = (a, b)$ will be denoted by $\Psi(a, b)$, and denote the set of all such a functions as Ψ :

$$\Psi \stackrel{\text{def}}{=} \bigcup_{1 \leq a < b \leq \infty} \Psi(a, b).$$

The Grand Lebesgue spaces (GLS) are rearrangement invariant, see [2], [7], [8], [10], and are used, for example, in the theory of probability [6], [15], [16]; theory of Partial Differential Equations [3], [4]; functional analysis [3], [4], [11], [16]; theory of Fourier series, theory of martingales, mathematical statistics, theory of approximation etc.

The so-called *fundamental function* $\phi(\delta) = \phi(G\psi(a, b), \delta)$, $\delta \in (0, \infty)$ of the $G\psi(a, b)$ space may be calculated by the formula

$$\phi(G\psi(a, b), \delta) = \sup_{p \in (a,b)} \left[\frac{\delta^{1/p}}{\psi(p)} \right].$$

This notion play a very important role in the theory of interpolation of operators, Fourier series [2], theory of random variables, in particular, theory of Central Limit Theorem, in Banach spaces [18] etc.

The detail investigation of fundamental functions for Grand Lebesgue Spaces with consideration of some examples see in the articles [11], [19].

Let us introduce also the so-called *localized* GLS norm. Indeed, we define for $\psi \in \Psi(a, b)$, $A : A \in \mathbf{B}$, $\mu(A) \in (0, \infty)$, and measurable function $f : A \rightarrow R$

$$\|f\|_{\psi, A} := \sup_{p \in (a, b)} \left[\frac{|f|_{p, A}}{\psi(p)} \right] = \sup_{p \in (a, b)} \left[\frac{[\int_A |f(x)|^p \mu(dx)]^{1/p}}{\psi(p)} \right]. \quad (1.4)$$

2 Main result.

Let ψ, ν be two function from the set Ψ .

Definition 2.1. The following functional $R(G\psi, G\nu) =$

$$R(\psi, \nu) \stackrel{def}{=} \sup_{0 \neq f \in G\psi \cap G\nu} \sup_{A: 0 < \mu(A) < \infty} \left[\frac{\|f\|_{\psi, A}}{\phi(G(\psi), \mu(A))} : \frac{\|f\|_{\nu, A}}{\phi(G(\nu), \mu(A))} \right] \quad (2.1)$$

will named *double ratio* between the spaces $G\psi$ and $G\nu$.

Theorem 2.1. Let ψ, ν be two function such that $\psi \in \Psi(a_1, b_1)$, $\nu \in \Psi(a_2, b_2)$. Suppose $b_1 < a_2$; the opposite case is trivial for us. Our statement:

$$R(G\psi, G\nu) = 1. \quad (2.2)$$

Proof.

First step. Let $f \in G\psi \cap G\nu$, $f \neq 0$, $p \in (a_1, b_1)$, $g \in (a_2, b_2)$, $0 < \mu(A) < \infty$. We rewrite the Lyapunov's inequality (2.1) as follows:

$$|f|_{p, A} \leq \frac{|f|_{q, A}}{\mu^{1/q}(A)} \cdot \mu^{1/p}(A),$$

or after dividing over $\psi(p)$

$$\frac{|f|_{p, A}}{\psi(p)} \leq \frac{|f|_{q, A}}{\mu^{1/q}(A)} \cdot \frac{\mu^{1/p}(A)}{\psi(p)}. \quad (2.3)$$

We have taking the supremum over p , $p \in (a_1, b_1)$ from both the sides of inequality (2.3) using the direct definition of the Grand Lebesgue norm and also the definition of the fundamental function

$$\|f\|_{\psi, A} \leq \frac{|f|_{q, A}}{\mu^{1/q}(A)} \cdot \phi(G\psi, \mu(A)). \quad (2.4)$$

Second step. Further, we use the simple estimate $|f|_{q,A} \leq \|f\|_{\nu,A} \cdot \nu(q)$, $q \in (a_2, b_2)$, in the inequality (2.4):

$$\frac{\|f\|_{\psi,A}}{\phi(G\psi, \mu(A))} \leq \|f\|_{\nu,A} \cdot \frac{\nu(q)}{\mu^{1/q}(A)} =$$

$$\|f\|_{\nu,A} \cdot \left[\frac{\mu^{1/q}(A)}{\nu(q)} \right]^{-1},$$

and after taking infimum over q :

$$\frac{\|f\|_{\psi,A}}{\phi(G\psi, \mu(A))} \leq \|f\|_{\nu,A} \cdot \inf_{q \in (a_2, b_2)} \left[\frac{\nu(q)}{\mu^{1/q}(A)} \right] =$$

$$\|f\|_{\nu,A} \cdot \left[\sup_{q \in (a_2, b_2)} \frac{\mu^{1/q}(A)}{\nu(q)} \right]^{-1} = \frac{\|f\|_{\nu,A}}{\phi(G\nu, \mu(A))},$$

hence $R(G\psi, G\nu) \leq 1$.

Lower bound. It is easy to verify that the lower bound for the expression for the functional $R(G\psi, G\nu)$ is attained, for example, if $f_0(x) = 1$, so that $R(G\psi, G\nu) \geq 1$. In detail,

$$\frac{\|f_0\|_{\psi,A}}{\phi(G\psi, \mu(A))} = \frac{\sup_{p \in (a_1, a_2)} \left\{ [\int_A 1 \cdot \mu(dx)]^{1/p} / \psi(p) \right\}}{\phi(G\psi, \mu(A))} =$$

$$\frac{\sup_{p \in (a_1, a_2)} \left\{ [\mu(A)]^{1/p} / \psi(p) \right\}}{\phi(G\psi, \mu(A))} = \frac{\phi(G\psi, \mu(A))}{\phi(G\psi, \mu(A))} = 1$$

and analogously $\|f_0\|_{\nu,A} / \phi(G\nu, A) = 1$.

This completes the proof of theorem 2.1.

Remark 2.1. The definition (2.1) may be easily extended on arbitrary pair of rearrangement invariant (r.i.) spaces F_1 and F_2 with norms correspondingly $\|f\|_{F_1}$, $\|f\|_{F_2}$, constructed over source triple (X, \mathbf{B}, μ) . Namely, denote

$$\|f\|_{F_1,A} = \|f \cdot I_A\|_{F_1}, \quad \|f\|_{F_2,A} = \|f \cdot I_A\|_{F_2},$$

where as usually $I_A = I_A(x)$, $x \in X$ is indicator function generated by the measurable set A . Then by definition of the double ratio between the spaces F_1 and F_2 is the following functional

$$R(F_1, F_2) \stackrel{def}{=} \sup_{0 \neq f \in F_1 \cap F_2} \sup_{A: 0 < \mu(A) < \infty} \left[\frac{\|f\|_{F_1,A}}{\phi(F_1, \mu(A))} : \frac{\|f\|_{F_2,A}}{\phi(F_2, \mu(A))} \right].$$

Obviously, $R(F_1, F_2) \geq 1$ for any pair of r.i. spaces F_1, F_2 .

Remark 2.2. If we introduce the *discontinuous* function

$$\psi_r(p) = 1, \quad p = r; \quad \psi_r(p) = \infty, \quad p \neq r, \quad p, r \in (a, b)$$

and define formally $C/\infty = 0$, $C = \text{const} \in R^1$, then the norm in the space $G(\psi_r)$ coincides with the L_r norm:

$$\|f\|_{G(\psi_r)} = |f|_r.$$

Thus, the inequality (1.2) is particular, more exactly, extremal case of the assertion of theorem 2.1.

3 Concluding remarks.

Our considerations are very similar to ones in the article [16], devoted to the generalization of Nilol'skii inequality, and may be used perhaps in the theory of operators acting in the Lorentz spaces, see, e.g. [1], [13], [14], [9], [12].

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